6.6 - Euler’s Method

Leonhard Euler made a huge number of contributions to mathematics, almost half after he was totally blind.

(When this portrait was made he had already lost most of the sight in his right eye.)

Leonhard Euler 1707 - 1783

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Greg Kelly, Hanford High School, Richland, Washington

It was Euler who originated the following notations:

$f(x)$ (function notation)

e (base of natural log)

π (pi)

i $\left(\sqrt{-1}\right)$

Σ (summation)

Δy (finite change)

Leonhard Euler 1707 - 1783

$\rightarrow$

There are many differential equations that can not be solved. We can still find an approximate solution.

Goal: to find an approximate solution curve for a differential equation that can't be solved exactly.

Notation update: $$\frac{dy}{dx} = f'(x, y)$$

Recall our Linearization formula (with new notation – just the equation of a tangent line through (x_0, y_0)):

$$L(x) = y = y_0 + f'(x_0, y_0)(x - x_0)$$

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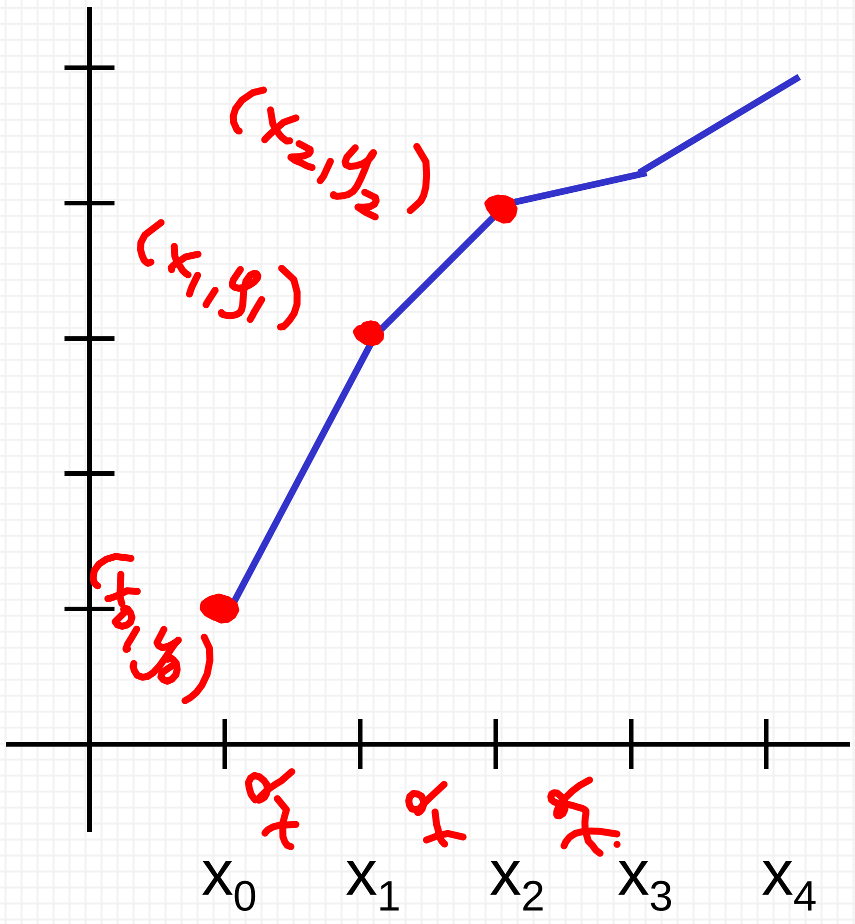
Logic: Starting with an initial point (that is given), use linearizations (tangent lines) to find subsequent points on the *approximate* solution curve.



$$y = f(x_0) + f'(x_0, y_0)(x - x_0)$$

$$y_1 = y_0 + f'(x_0, y_0)(x_1 - x_0)$$

$$y_1 = y_0 + f'(x_0, y_0)dx$$

$$y_2 = y_1 + f'(x_1, y_1)dx$$

$$y_3 = y_2 + f'(x_2, y_2)\,dx$$

$$\boxed{y_{n+1} = y_n + f'(x_n, y_n)dx}$$

The procedure is known as **Euler's Method**

Example: $\frac{dy}{dx} = 1 + y, \ \ y(0) = 1, \quad dx = .1$

$y_1 = y_0 + f'(x_0, y_0)\,dx$

$= 1 + (1 + 1)(.1) = 1.2$

$y_2 = y_1 + f'(x_1, y_1)\,dx$

$= 1.2 + (1 + 1.2)(.1) = 1.42$

$$\begin{array}{r} 1.42 \\ +\ .242 \\ \hline \end{array}$$

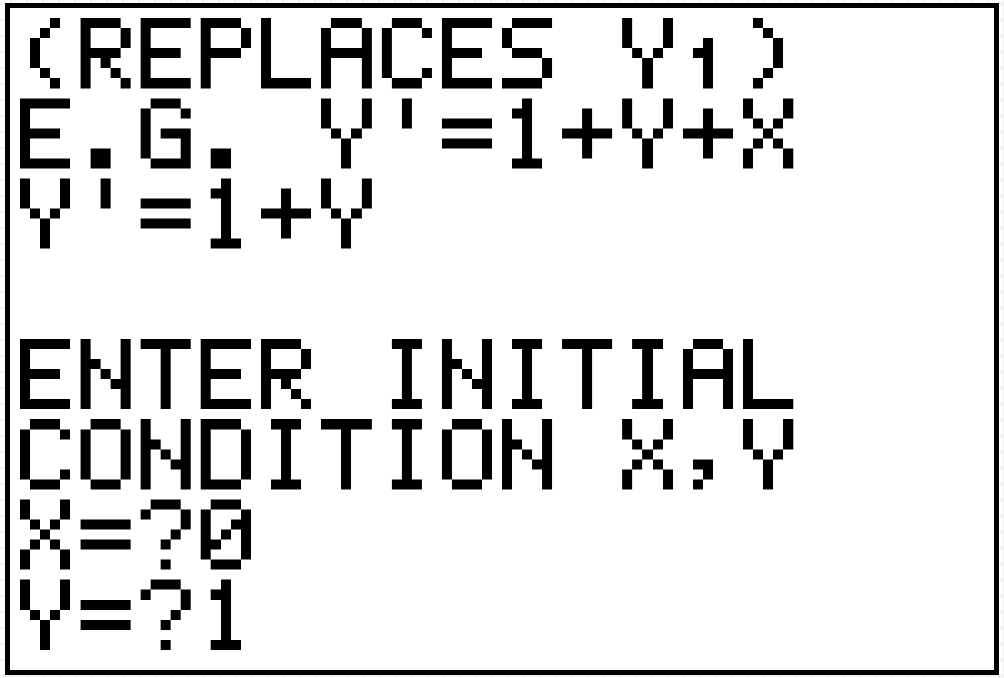
$y_3 = y_2 + f'(x_2, y_2)\,dx$

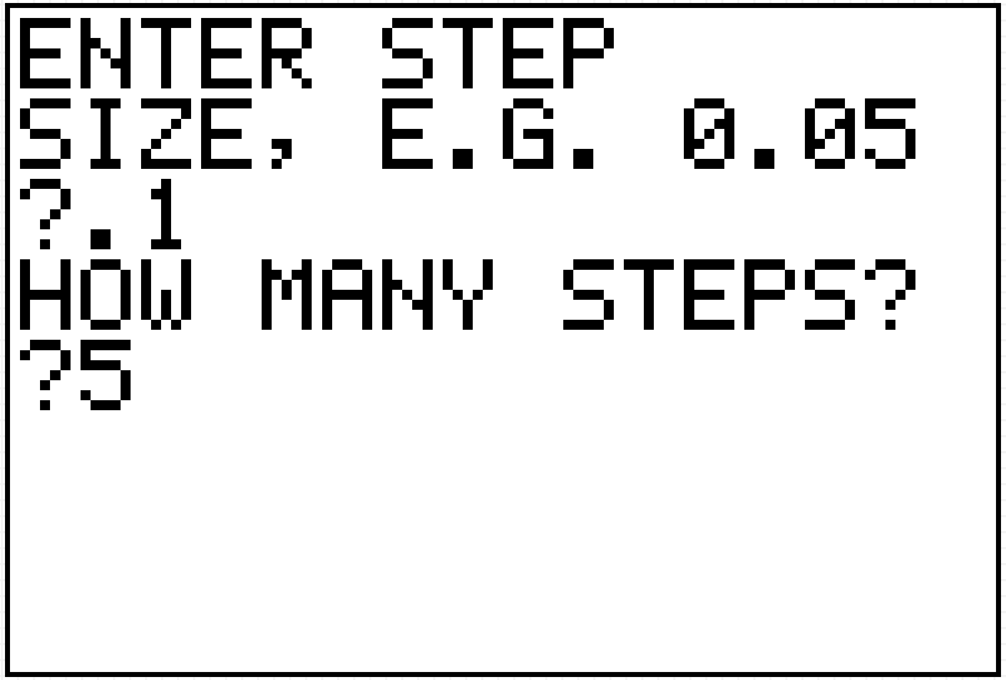
$= 1.42 + (1 + 1.42)(.1)$

x	y
x_0 0	y_0 1
x_1 .1	y_1 1.2
x_2 .2	y_2 1.42
x_3 .3	y_3 1.662

$f(.3) \approx 1.662$

The program EULER will generate the entire table for a specified number of steps – try it now for the same problem with 5 steps.





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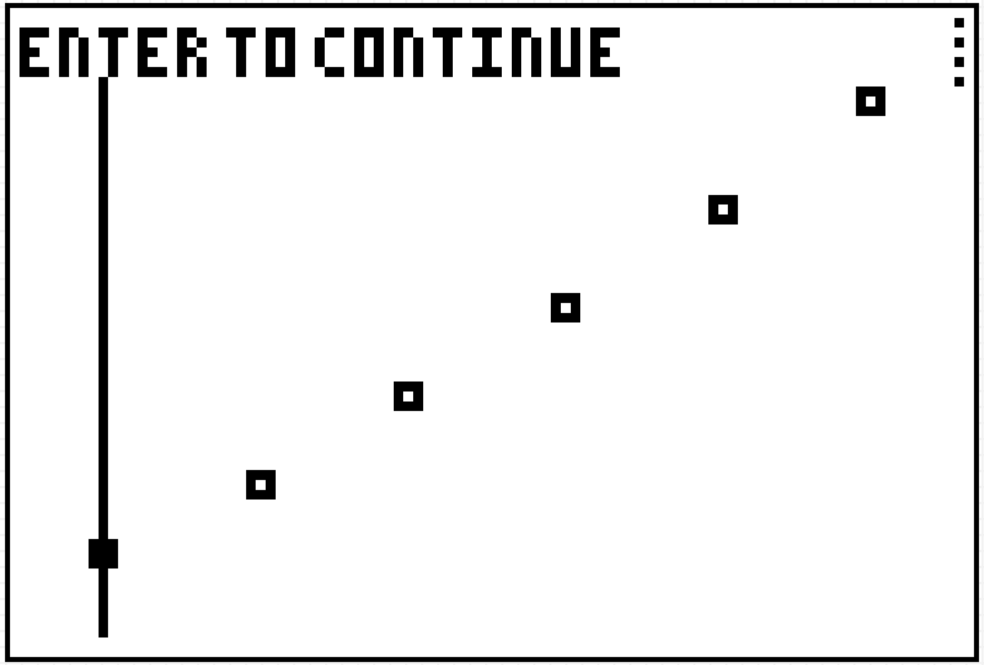

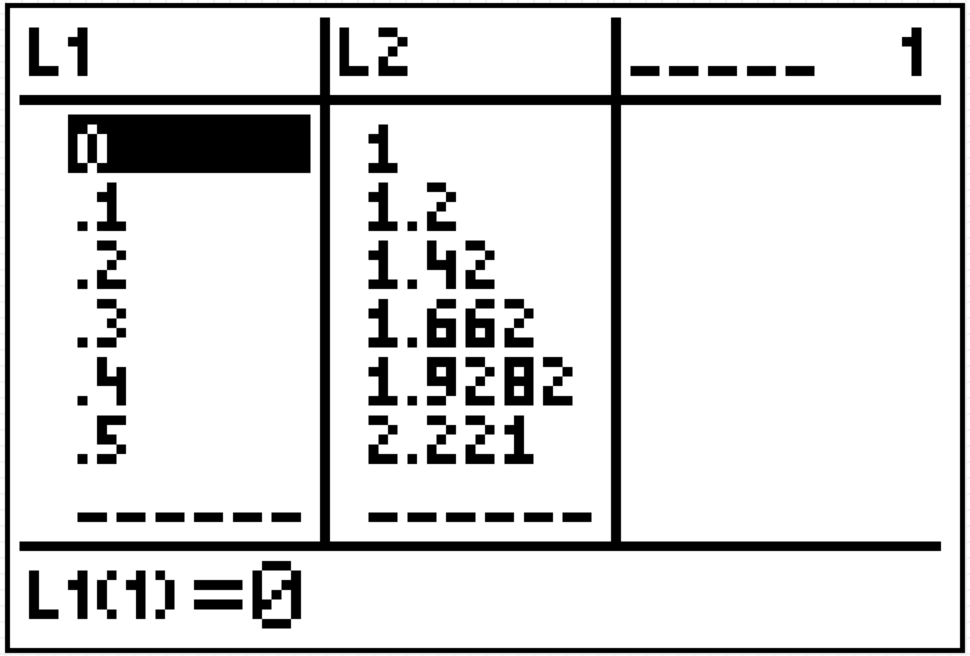

L1	L2	------ 1
0	1	
.1	1.2	
.2	1.42	
.3	1.662	
.4	1.9282	
.5	2.221	
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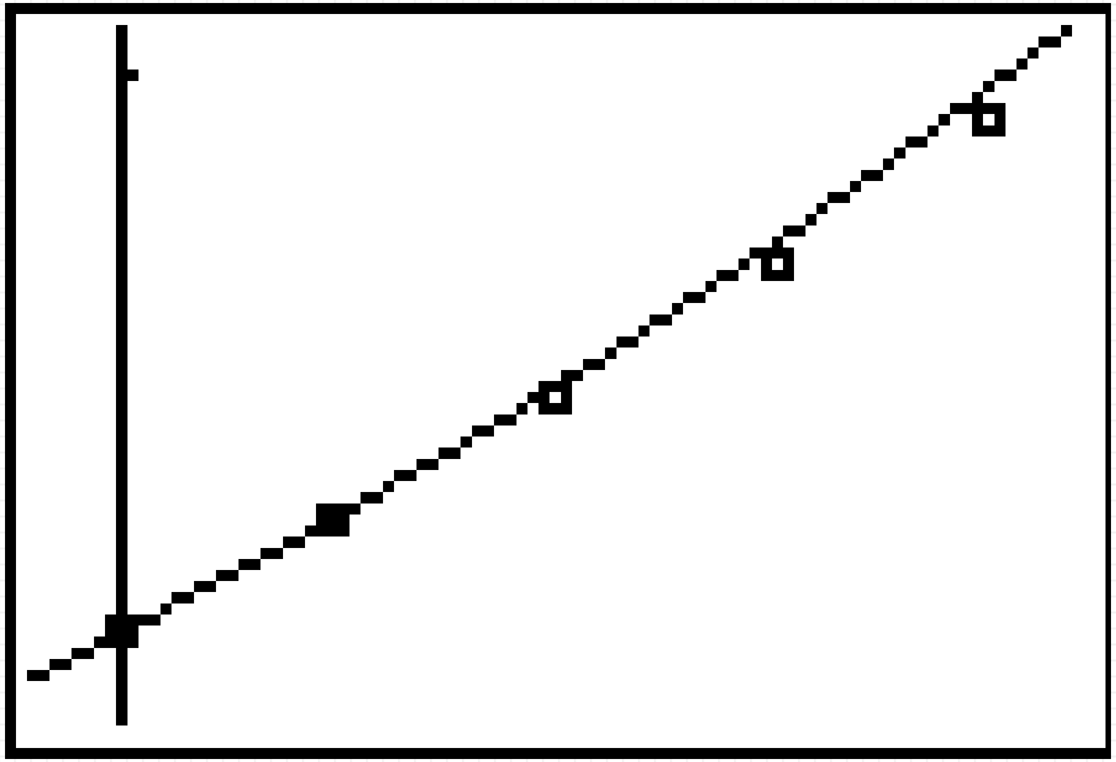
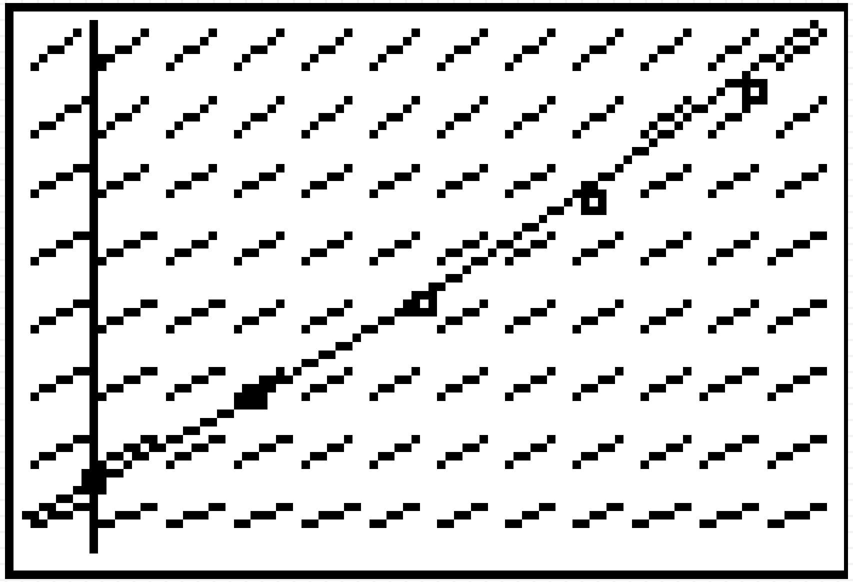
L1(1)=0

The actual solution curve is $y = 2e^x - 1$. Graph this along with the approximate solution from EULER.



Draw the slope field for the same differential equation to see how the approximate solution and the actual solution compare to the slope field (keep the same window from when you used EULER.)



Example: Approximate $f(-1)$ using Euler's Method and two steps of equal size.

$dx = -\frac{1}{2}$

STEP SIZE $= -1/2$ (dx)

$$\frac{dy}{dx} = 2x - 3y, \;\; y(0) = 1$$

$y_1 = y_0 + f'(x_0, y_0)\,dx$

$= 1 + (2\cdot 0 - 3\cdot 1)(-\frac{1}{2})$

$= 1 + (-3)(-1/2) = 5/2$

$y_2 = y_1 + f'(x_1, y_1)\,dx$

$= \frac{5}{2} + (2\cdot\frac{-1}{2} - 3\cdot\frac{5}{2})(-\frac{1}{2})$

$= \frac{5}{2} + (\frac{-17}{2})(\frac{-1}{2}) = \frac{5}{2} + \frac{17}{4}$

$$\boxed{f(-1) \approx \frac{27}{4}}$$

	x		y
x_0	0	y_0	1
x_1	$-\frac{1}{2}$	y_1	$5/2$
x_2	-1	y_2	$\boxed{\frac{27}{4} \approx f(-1)}$

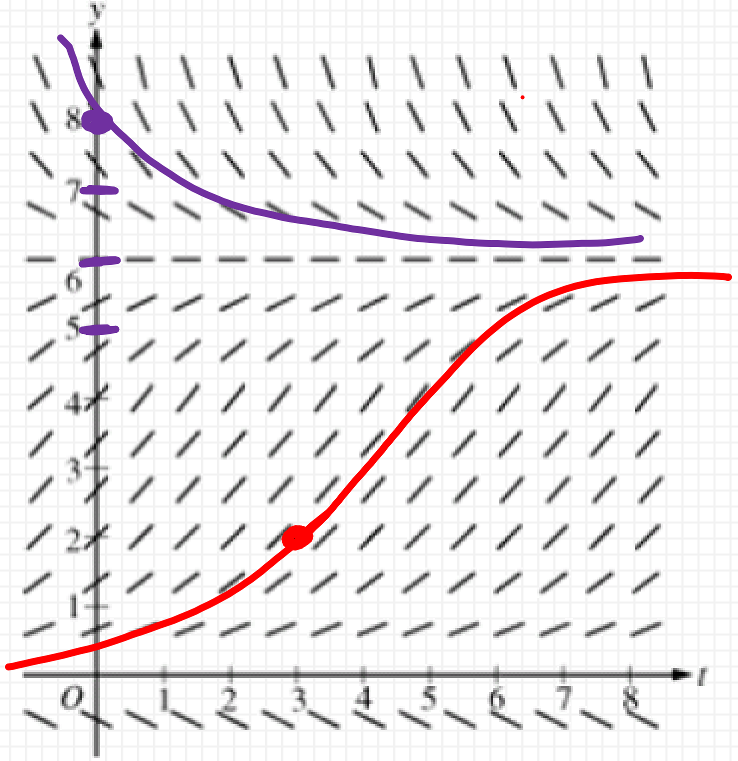
2008 SCORING GUIDELINES

Question 6

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.
(Note: Use the axes provided in the exam booklet.)

(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.



$$y_1 = y_0 + f'(x_0, y_0)\,dx$$
$$= 8 + \left(\frac{8}{8}(6-8)\right)\left(\frac{1}{2}\right) = 7$$

$$y_2 = y_1 + f'(x_1, y_1)\,dx$$
$$= 7 + \left(\frac{7}{8}(6-7)\right)\left(\frac{1}{2}\right)$$
$$= 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right)$$
$$= 7 - \frac{7}{16} = 6\frac{9}{16}$$

x	y
0	8
1/2	7
1	$6\frac{9}{16} \approx f(1)$

2009 SCORING GUIDELINES

Question 4

Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

Homework:

AP Packet #65 – 67